TZITZÉICA TRANSFORMATION IS A DRESSING ACTION

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Abstract. We classify the simplest rational elements in a twisted loop group, and prove that dressing actions of them on proper indefinite affine spheres give the classical Tzitzéica transformation and its dual. We also give the group point of view of the Permutability Theorem, construct complex Tzitzéica transformations, and discuss the group structure for these transformations.

1. Introduction

In 1910, Tzitzéica published a classical paper [21] on hyperbolic surfaces in $\mathbb{R}^3$ whose Gauss curvature at any point p is proportional to the fourth power of the distance from a fixed point to the tangent plane at p. He proved

$$w_{xy} = e^w - e^{-2w} \tag{1.1}$$

is the structure equation, and also constructed a geometric transformation of such surfaces that is similar to the well-known Bäcklund transformation of surfaces with constant negative curvature. These surfaces are invariant under affine transformations, and they are now known as (proper) affine spheres in affine differential geometry.

The classical Tzitzéica equation (1.1) was rediscovered in many mathematical and physical contexts afterwards (see, e.g., [5], [7], [8]). In recent years, techniques from soliton theory have been applied to this equation extensively by, e.g.: Rogers & Schief in the context of gas dynamics ([15]), Kaptsov & Shan'ko on multi-soliton formulas ([11]), Dorfmeister & Eitner on Weierstrass type representation ([6]), and Bobenko & Schief on its discretizations ([3], [4]). Terng & Uhlenbeck ([19]) gave a systematic method to construct Bäcklund-type transformations via dressing actions of simple rational loop group elements. It is natural to ask whether the classical Tzitzéica transformation is a dressing action of some loop element, whether there are new transformations of affine spheres, and what is the group structure of these transformations. This paper answers these questions.

In section 2, we give a brief review of classical results and provide the Lax pair of the structure equations. In section 3, we review the reality conditions for this Lax pair and give the loop group description of indefinite affine spheres. We then classify the simplest rational elements in this loop group and compute their dressing actions on affine spheres in section 4. These rational elements cannot be constructed by projections as in [19] and the computation is harder. It turns out that one class of dressing action provides exactly the Tzitzéica transformation and the other provides the dual transformation. In section 5, we present the group point of view of the classical permutability theorem, construct complex Tzitzéica transformations and discuss the group structure of these transformations. Some examples are presented in the last section.

2. Indefinite affine sphere and its Lax representation

Classical affine differential geometry studies the properties of surfaces in $\mathbb{R}^3$ invariant under the (equi-)affine transformations $x \to Ax + v$, where $A \in \mathrm{SL}(3,\mathbb{R})$ and $x,\ v \in \mathbb{R}^3$. There are three fundamental affine invariants: the affine (or Blaschke) metric, the Fubini-Pick cubic form, and the third fundamental form (or the affine shape operator). These invariants satisfy certain compatibility equations and the Fundamental Theorem states that they then determine a surface uniquely up to affine transformations. Let us first review the definitions of these invariants (for more details see, e.g., [2], [13], [18]). The reader may also refer to [2], [3], [4] or [17] for an elementary description of affine spheres.

Let $X : M \hookrightarrow \mathbb{R}^3$ be an immersed surface with non-degenerate second fundamental form. Let $E = (e_1, e_2, e_3)$ be a local $\mathrm{SL}(3,\mathbb{R})$-frame on M such that e_1, e_2 are tangent to M, and e_3 is transversal to M. Let ω_1, ω_2 denote the dual coframe of e_1, e_2, i.e.,

$$\mathrm{d}X = e_1 \otimes \omega_1 + e_2 \otimes \omega_2.$$

Let (ω_{AB}) denote the $\mathrm{sl}(3,\mathbb{R})$-valued 1-form $E^{-1}\,\mathrm{d}E$, i.e.,

$$\mathrm{d}e_A = \sum_{B=1}^{3} e_B \otimes \omega_{BA}.$$

Then we have the structure equation:

$$\begin{cases} \mathrm{d}\omega_A &= -\sum_B \omega_{AB} \wedge \omega_B, \\ \mathrm{d}\omega_{AB} &= -\sum_C \omega_{AC} \wedge \omega_{CB}. \end{cases} \tag{2.1}$$

Since $\omega_3 = 0$ on M, (2.1) implies that for $i = 1, 2$:

$$\omega_{3,i} = h_{i1}\omega_1 + h_{i2}\omega_2, \text{ with } h_{ij} = h_{ji}. \tag{2.2}$$

A direct computation shows that the quadratic form

$$g := |\det(h_{ij})|^{-\frac{1}{4}} \sum_{i,j=1}^{2} h_{ij}\omega_i\omega_j \tag{2.3}$$

is invariant under change of affine frames, and it is called the *affine metric* of M. M is said to be *definite* or *indefinite* if the affine metric is definite or indefinite respectively.

The *affine normal* is $\xi := \triangle X \,/\, 2$, where $\triangle$ is the Laplacian of g. It satisfies two natural geometric conditions:

(i) $\mathrm{d}\xi(\cdot) \in \mathrm{T}M$,
(ii) $i_\xi\,\mathrm{d}V = dvol_g$ (the volume form of g);

and is essentially determined by them.

Take the exterior differentiation of (2.2) to get

$$\sum_j (\mathrm{d}h_{ij} + h_{ij}\omega_{3,3} - h_{ik}\omega_{kj} - h_{kj}\omega_{ki}) \wedge \omega^j = 0, \tag{2.4}$$

and define h_{ijk} by

$$\sum_k h_{ijk}\omega_k = \mathrm{d}h_{ij} + h_{ij}\omega_{3,3} - h_{ik}\omega_{kj} - h_{kj}\omega_{ki}. \tag{2.5}$$

Then (2.2) and (2.4) imply that h_{ijk} is symmetric in i, j, k. *The Fubini-Pick cubic form* is defined as

$$J := \sum_{i,j,k} h_{ijk}\omega_i\omega_j\omega_k,$$

which is an affine invariant.

We choose $e_3 = \xi$. Then $\omega_{3,3} = 0$. Exterior differentiate it to get

$$\omega_{13} \wedge \omega_{31} + \omega_{23} \wedge \omega_{32} = 0.$$

Thus the following form

$$\mathrm{I\!I\!I} := |\det(h_{ij})|^{\frac{1}{4}}(\omega_{13}\omega_{31} + \omega_{23}\omega_{32})$$

is symmetric. This is the *third fundamental form.* Equivalently, we can first define the *affine shape operator* S:

$$S(u) := \mathrm{d}\xi(u), \quad \forall\, u \in \mathrm{T}_pM,$$

then $\mathrm{I\!I\!I}\,(u, v) = g(S(u), v) = g(u, S(v))$. The *affine mean curvature* H and the *affine Gauss curvature* K are defined as $H = \mathrm{Tr}\, S\,/\,2,\ K = \det S$.

Definition 2.1. An *affine sphere* is a surface all of whose affine normals meet at a common point.

An equivalent definition is $S = H \cdot \mathrm{Id}$, i.e., the shape operator is a scalar multiple of the identity map and H is then the affine mean curvature. It follows from the structure equations that H must be constant. When $H = 0$, all affine normals are parallel and the center is at infinity. Such surface is called *improper affine sphere* and has been completely classified in [2]. When $H \neq 0$, it is called *proper affine sphere* and we can move the center to the origin and normalize H to 1 by scaling the ambient space and changing the orientation if necessary. Then $e_3 = \xi = X$.

From now on we will only consider proper indefinite affine spheres in $\mathbb{R}^3$ with $\xi = X$. First note that there exists local asymptotic coordinate

system (x, y) and a smooth function $w(x, y)$ such that the affine metric is:

$$g = e^{w}(\,\mathrm{d}x \otimes \,\mathrm{d}y + \,\mathrm{d}y \otimes \,\mathrm{d}x).$$

We choose a frame $e_1 = X_x$, $e_2 = e^{-w}X_y$, and $e_3 = \xi = X$. Then $\det(e_1, e_2, e_3) = 1$, and

$$\begin{aligned} &\omega_1 = \,\mathrm{d}x = \omega_{13} = \omega_{32}, && \omega_2 = e^{w}\,\mathrm{d}y = \omega_{23} = \omega_{31}, \\ &\omega_{33} = 0, && (h_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

A direct computation using the formula (2.5) shows

$$\begin{cases} \omega_{21} = a\,\mathrm{d}x,\ \omega_{12} = be^{-2w}\,\mathrm{d}y & \text{for some functions } a, b; \\ J = -2a\,\mathrm{d}x^3 - 2b\,\mathrm{d}y^3. \end{cases}$$

Finally from $\,\mathrm{d}\omega_A + \sum_B \omega_{AB} \wedge \omega_B = 0$ we get

$$\omega_{11} = -\omega_{22} = w_x\,\mathrm{d}x.$$

We have obtained the flat $\mathrm{sl}(3, \mathbb{R})$-valued 1-form

$$\omega = E^{-1}\,\mathrm{d}E = \begin{pmatrix} w_x\,\mathrm{d}x & be^{-2w}\,\mathrm{d}y & \mathrm{d}x \\ a\,\mathrm{d}x & -w_x\,\mathrm{d}x & e^{w}\,\mathrm{d}y \\ e^{w}\,\mathrm{d}y & \mathrm{d}x & 0 \end{pmatrix}.$$

The compatibility equations are

$$\mathrm{d}\omega + \omega \wedge \omega = 0 \iff \begin{cases} w_{xy} = e^{w} - ab\ e^{-2w}, \\ a_y = 0, \quad b_x = 0. \end{cases} \tag{2.6}$$

When $ab \neq 0$ we may reparametrize the asymptotic coordinates to make $a = b = 1$. Then (2.6) is simplified to the classical Tzitzéica equation.

Remark 2.2. It is known that ruled proper indefinite affine spheres correspond to the case $ab \equiv 0$ and they have been well understood (see [13]). For non-ruled case, the points at which $ab = 0$ are called *planar points.*

The following observation is crucial for the integrability of proper indefinite affine spheres: The system (2.6) is invariant under the transformation

$$a \longrightarrow \gamma a, \quad b \longrightarrow \gamma^{-1} b$$

with $\gamma \in \mathbb{C} \setminus \{0\}$. Thus a family of flat connections is obtained:

$$\omega_\gamma = \begin{pmatrix} w_x\,\mathrm{d}x & \gamma^{-1}be^{-2w}\,\mathrm{d}y & \mathrm{d}x \\ \gamma a\,\mathrm{d}x & -w_x\,\mathrm{d}x & e^{w}\,\mathrm{d}y \\ e^{w}\,\mathrm{d}y & \mathrm{d}x & 0 \end{pmatrix}.$$

The zero curvature equation of ω_γ is called the *Lax representation* of (2.6).

When we solve E_γ from

$$E_\gamma^{-1}\,\mathrm{d}E_\gamma = \omega_\gamma \tag{2.7}$$

for any $\gamma \in \mathbb{R}\setminus\{0\}$, the last column of E_γ gives a family of affine spheres, whose affine fundamental invariants are:

$$g = 2e^w\,\mathrm{d}x\,\mathrm{d}y, \qquad S = \mathrm{Id}, \qquad J = -2\gamma a\ \mathrm{d}x^3 - \frac{2}{\gamma}b\ \mathrm{d}y^3.$$

Let us recall the classical *duality relation* for indefinite affine spheres. Let $h = e^\omega$. Then Tzitzéica equation becomes

$$(\ln h)_{xy} = h - \frac{1}{h^2}, \qquad \text{or}\ \ h_{xy}h - h_x h_y = h^3 - 1. \tag{2.8}$$

Classically (2.7) with $a = b = 1$ was written as a linear system for X:

$$\begin{cases} X_{xx} = \frac{h_x}{h}X_x + \frac{\gamma}{h}X_y\ , \\ X_{xy} = hX\ , \\ X_{yy} = \frac{1}{\gamma h}X_x + \frac{h_y}{h}X_y\ . \end{cases} \tag{2.9}$$

If X solves (2.9), then we can check that

$$X^* := \frac{1}{h}X_x \times X_y$$

is a solution of (2.9) with γ replaced by $-\gamma$. Here $\times$ is the vector cross product in $\mathbb{R}^3$. This is clearly a duality relation: $(X^*)^* = X$.

Finally let us recall the classical Tzitzéica transformation:

Theorem 2.3 ([21])**.** *Given a solution (h, X) of (2.8) and (2.9), and ϕ_1 any scalar solution of (2.9) with parameter γ_1, then the following transformation produces a new solution (h_1, X_1) of (2.8) and (2.9):*

$$\begin{cases} h_1 := h - 2(\ln\phi_1)_{xy}\ , \\ X_1 := \dfrac{(\gamma-\gamma_1)hX - 2\gamma(\ln\phi_1)_x X_y + 2\gamma_1(\ln\phi_1)_y X_x}{(\gamma+\gamma_1)h}\ . \end{cases} \tag{2.10}$$

3. The reality conditions and loop group description

Henceforth we assume $a = b = 1$. To further reveal the hidden symmetry, let $\lambda = \sqrt[3]{\gamma}$ and change the frame E_γ to

$$F_\lambda = E_\gamma\,\mathrm{diag}(1/\lambda, \lambda, 1).$$

The gauged family of flat connections is then

$$\theta_\lambda = F_\lambda^{-1}\,\mathrm{d}F_\lambda = \begin{pmatrix} w_x & 0 & \lambda \\ \lambda & -w_x & 0 \\ 0 & \lambda & 0 \end{pmatrix} \mathrm{d}x + \lambda^{-1} \begin{pmatrix} 0 & e^{-2w} & 0 \\ 0 & 0 & e^w \\ e^w & 0 & 0 \end{pmatrix} \mathrm{d}y. \tag{3.1}$$

For any $g \in \mathrm{SL}(3,\mathbb{C})$, we need to define $\tau(g) := \bar{g}$ and define σ by:

$$\sigma(g) := T\ (g^t)^{-1}T^{-1}, \qquad \text{where } T = \begin{pmatrix} 0 & 1 & 0 \\ -\epsilon & 0 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}, \quad \epsilon = e^{\pi \mathrm{i}/3}.$$

The automorphism σ has order 6 and induces the following automorphism (still denoted by σ) on the Lie algebra $\mathrm{sl}(3,\mathbb{C})$: $\sigma(A) = -TA^tT^{-1}$. Therefore σ gives the eigenspace decomposition: $\mathrm{sl}(3,\mathbb{C}) = \oplus_{j=0}^5 \mathcal{G}_j$, where $\mathcal{G}_j$ is of eigenvalue ϵ^j. We compute that $X_j \in \mathcal{G}_j$ if and only if

$$X_0 = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & -x_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 & x_{13} \\ x_{21} & 0 & 0 \\ 0 & x_{13} & 0 \end{pmatrix},$$
$$X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23} \\ -x_{23} & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & -2x_{11} \end{pmatrix},$$
$$X_4 = \begin{pmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & -x_{13} & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & x_{12} & 0 \\ 0 & 0 & x_{23} \\ x_{23} & 0 & 0 \end{pmatrix}.$$

Note that $\sigma\tau = \tau^{-1}\sigma^{-1}$ implies that $\oplus_{j=0}^5(\mathrm{sl}(3,\mathbb{R}) \cap \mathcal{G}_j)$ is the corresponding eigenspace decomposition of $\mathrm{sl}(3,\mathbb{R})$.

The θ_λ in (3.1) satisfies two reality conditions (first given in [12]):

$$\tau(\theta_\lambda) = \theta_{\bar{\lambda}}, \qquad \sigma(\theta_\lambda) = \theta_{\epsilon\lambda}. \tag{3.2}$$

When we solve F_λ in (3.1) uniquely with the initial condition $F(0,0,\lambda) = I$, it is easy to show that F_λ also satisfies the reality conditions (3.2).

Let $\mathbb{C}_* := \mathbb{C}\setminus\{0\}$. We adopt the following notations for loop groups:

$$\begin{aligned} \Lambda G &= \{ \text{ holomorphic maps from } \mathbb{C}_* \cap (\mathcal{O}_r \cup \mathcal{O}_{1/r}) \text{ to } G \}, \\ \Lambda_+ G &= \{ \text{ holomorphic maps from } \mathbb{C}_* \text{ to } G \}, \\ \Lambda_- G &= \{ \text{ holomorphic maps } f \text{ from } \mathcal{O}_r \cup \mathcal{O}_{1/r} \text{ to } G \text{ with } f(\infty) = I \}, \end{aligned}$$

where $0 < r < 1$ is sufficiently small and

$$\mathcal{O}_r = \{\lambda \in \mathbb{C} \ : \ |\lambda| < r\}, \qquad \mathcal{O}_{1/r} = \{\lambda \in \mathbb{C} \cup \{\infty\} \ : \ |\lambda| > 1/r\}\ .$$

Similar notations also apply to their Lie algebras. Let $\Lambda^{\tau,\sigma}G$ denote the subgroup of $g \in \Lambda G$ satisfying the reality conditions (3.2). Then θ_λ in (3.1) is a $\Lambda_+^{\tau,\sigma}\,\mathrm{sl}(3,\mathbb{C})$-valued flat connection, and the corresponding frame F_λ for indefinite affine spheres lies in $\Lambda_+^{\tau,\sigma}\,\mathrm{SL}(3,\mathbb{C})$. Conversely, given any smooth map F from a domain in $\mathbb{R}^2$ to $\Lambda_+^{\tau,\sigma}\,\mathrm{SL}(3,\mathbb{C})$ satisfying

$$F^{-1}F_x = A\lambda + B, \qquad F^{-1}F_y = C\lambda^{-1} + D \tag{3.3}$$

with $A_{32}C_{31} \neq 0$, the last column of F then gives an affine sphere with $h = A_{32}C_{31}$ and F differs from F_λ in (3.1) by a simple gauge. This is the loop group description for indefinite affine spheres (for details, see [3], [6]).

4. Dressing actions of simple rational elements

Let us briefly review the method of dressing action (the original idea went back to [22] but see [9] or [20] for an elementary introduction). Let $G = \mathrm{SL}(3,\mathbb{C})$, $g(\lambda) \in \Lambda_-^{\tau,\sigma}G$, and $F(x,y,\lambda) \in \Lambda_+^{\tau,\sigma}G$ the frame of an associated family of indefinite affine spheres. Assume we can do the following factorization for each fixed (x,y):

$$g(\lambda)\ F(x,y,\lambda) = \tilde{F}(x,y,\lambda)\ \tilde{g}(x,y,\lambda)\ , \tag{4.1}$$

with $\tilde{F} \in \Lambda_+^{\tau,\sigma}G$ and $\tilde{g} \in \Lambda_-^{\tau,\sigma}G$. Then $\tilde{F}$ also satisfies (3.3) and generates new affine spheres. We sketch the proof here. It suffices to prove that $\tilde{F}^{-1}(\tilde{F})_x$ and $\tilde{F}^{-1}(\tilde{F})_y$ are linear in λ and λ^{-1} respectively. But

$$\begin{aligned} \tilde{F}^{-1}(\tilde{F})_x &= \tilde{g}F^{-1}g^{-1}(gF\tilde{g}^{-1})_x \\ &= \tilde{g}(F^{-1}F_x)\tilde{g}^{-1} + \tilde{g}(\tilde{g}^{-1})_x \\ &= \tilde{g}(A\lambda + B)\tilde{g}^{-1} - \tilde{g}_x\tilde{g}^{-1}. \end{aligned}$$

On the left hand side, $\tilde{F}^{-1}(\tilde{F})_x$ is holomorphic in $\lambda \in \mathbb{C} \setminus \{0\}$; on the right it has a simple pole at ∞ since $g(\infty) = \tilde{g}(\infty) = I$. So

$$\tilde{F}^{-1}(\tilde{F})_x = \widetilde{A}\lambda + \tilde{B}.$$

Similarly, $\tilde{F}^{-1}(\tilde{F})_y$ is linear in $1/\lambda$. This completes the proof.

Furthermore, $g * F := \tilde{F}$ defines a group action of $\Lambda_-^{\tau,\sigma}G$ on the frames of affine spheres, which is called the *dressing action*.

The factorization (4.1) can indeed be done on a dense open subset of $\Lambda^{\tau,\sigma}G$ (see [1], [14]). There is no explicit construction for general g, but when g is a rational element, the factorization can be carried out using residue calculus (see [19]). In search of simple rational elements

in $\Lambda_-^{\tau,\sigma}\,\mathrm{SL}(3,\mathbb{C})$, it helps to write $\sigma = \nu\circ\mu$ as the composition of two commuting automorphisms, where

$$\begin{aligned} \nu(g) &:= QgQ^{-1} \quad \text{with } Q = \mathrm{diag}(\epsilon^4,\epsilon^2,1),\\ \mu(g) &:= P(g^t)^{-1}P \quad \text{with } P = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}. \end{aligned}$$

Here ν has been called the Coxeter-Killing automorphism, and the involution μ is the unique outer automorphism of $\mathrm{SL}(3,\mathbb{C})$ modulo inner ones. We observe that an element $g(\lambda)\in\Lambda G$ lies in $\Lambda^{\tau,\sigma}G$ if and only if

$$\tau(g_{\bar\lambda}) = g_\lambda, \qquad \nu(g_\lambda) = g_{\epsilon^4\lambda}, \qquad \mu(g_\lambda) = g_{-\lambda}. \tag{4.2}$$

Remark 4.1. τ and μ define a symmetric space $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$. Here $\mathrm{SO}(2,1)$ is the isometry group of the quadratic form given by the symmetric matrix P, i.e., $2x_1x_2+x_3^2$ on $\mathbb{R}^3$. It was proved in [20] that if F_λ is the frame of an associated family of indefinite affine spheres, then $F_{-1}F_1^{-1}$ is a harmonic map from $\mathbb{R}^{1,1}$ to the symmetric space $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$.

We will first study rational elements in $\Lambda_-^\sigma G$. Due to the ν-reality condition in (4.2), the simplest rational element in $\Lambda_-^\sigma G$ may have only 3 simple poles $\{\alpha,\epsilon^2\alpha,\epsilon^4\alpha\}$ with $\alpha\in\mathbb{C}_*$. The element can always take the following special form:

$$g(\lambda) = I + \frac{2\alpha}{\lambda-\alpha}A + \frac{2\epsilon^2\alpha}{\lambda-\epsilon^2\alpha}B + \frac{2\epsilon^4\alpha}{\lambda-\epsilon^4\alpha}C. \tag{4.3}$$

Plug it into (ν,μ)-reality conditions in (4.2) and compare the residues at each pole, we obtain that $g\in\Lambda_-^\sigma G$ if and only if

$$\begin{cases} B = Q^{-1}AQ, \qquad C = QAQ^{-1},\\ A^tP(I-A-2\epsilon B+2\epsilon^2C) = 0. \end{cases} \tag{4.4}$$

Write $A=(a_{ij})$, and we compute that

$$(I-A-2\epsilon B+2\epsilon^2C) = \begin{pmatrix} 1-3a_{11} & -3a_{12} & 3a_{13}\\ 3a_{21} & 1-3a_{22} & -3a_{23}\\ -3a_{31} & 3a_{32} & 1-3a_{33}\end{pmatrix}. \tag{4.5}$$

If the rank of A is 3, we get $A=I/3$ and $g(\lambda)$ is trivial. So there are two types left for A: rank 1 type and rank 2 type. A long but not hard computation implies that the **rank** 1 **type** is as follows:

$$A = \frac{1}{3}\begin{pmatrix}\frac{b}{2ab-1}\\ a\\ 1\end{pmatrix}\begin{pmatrix}a & b & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}\frac{ab}{2ab-1} & \frac{b^2}{2ab-1} & \frac{b}{2ab-1}\\ a^2 & ab & a\\ a & b & 1\end{pmatrix}, \tag{4.6}$$

with the corresponding loop group element

$$g(\lambda) = I + \frac{2}{\lambda^3 - \alpha^3}\begin{pmatrix} \frac{\alpha^3 ab}{2ab-1} & \frac{\alpha\lambda^2 b^2}{2ab-1} & \frac{\alpha^2\lambda b}{2ab-1} \\ \alpha^2\lambda a^2 & \alpha^3 ab & \alpha\lambda^2 a \\ \alpha\lambda^2 a & \alpha^2\lambda b & \alpha^3 \end{pmatrix}; \tag{4.7}$$

and the **rank 2 type** (*thus the matrix* (4.5) *has rank* 1) is as follows:

$$A = \frac{1}{3}\begin{pmatrix} \frac{ab-1}{2ab-1} & \frac{-b^2}{2ab-1} & \frac{b}{2ab-1} \\ a^2 & 1-ab & -a \\ -a & b & 0 \end{pmatrix}, \tag{4.8}$$

with the corresponding loop group element

$$g(\lambda) = I + \frac{2}{\lambda^3 - \alpha^3}\begin{pmatrix} \frac{\alpha^3(ab-1)}{2ab-1} & \frac{-\alpha\lambda^2 b^2}{2ab-1} & \frac{\alpha^2\lambda b}{2ab-1} \\ \alpha^2\lambda a^2 & \alpha^3(1-ab) & -\alpha\lambda^2 a \\ -\alpha\lambda^2 a & \alpha^2\lambda b & 0 \end{pmatrix}. \tag{4.9}$$

Both types must meet the constraint: $2ab \neq 1$.

Remark 4.2. We compute that $\det g$ is $[(\lambda^3+\alpha^3)/(\lambda^3-\alpha^3)]^{\mathrm{rank}(A)}$, i.e., only depending on the poles and the rank of the residues. A scaling by $(\det g)^{-1/3}$ will make them lie in $\mathrm{SL}(3,\mathbb{C})$, though not rational any more. Since the scaling does not affect the factorization and the dressing action, we will ignore this step henceforth.

Let $l := (a, b, 1)$, ℓ be the line $\mathbb{C}\cdot l$, and introduce the following 'cone':

$$\Delta := \{\, (z_1, z_2, z_3) \in \mathbb{C}^3 \mid 2z_1 z_2 = z_3^2, \text{ or } z_3 = 0 \,\}.$$

Then $2ab \neq 1$ is equivalent to $\ell \nsubseteq \Delta$. We observe that $\ell^t = \mathrm{Image}(\mathrm{Res}_\alpha\, g^t)$ for rank 1 type and $\ell^t = \mathrm{Kernel}(\mathrm{Res}_\alpha\, g\, P)$ for rank 2 type (here ℓ^t means $\mathbb{C}\cdot l^t$). Conversely, a line ℓ not in Δ determines the residue A and thus the simple element g uniquely in both types. Henceforth we always use $g_{\alpha,\ell}$ to denote the rank 1 type element (4.7) and use $h_{\alpha,\ell}$ to denote the rank 2 type element (4.9). We have proved the following theorem:

Theorem 4.3. *The simplest rational element in* $\Lambda_-^\sigma\,\mathrm{GL}(3,\mathbb{C})$ *is either* $g_{\alpha,\ell}$ *of rank* 1 *type* (4.7) *or* $h_{\alpha,\ell}$ *of rank* 2 *type* (4.9), *where* $\alpha \in \mathbb{C}_*$ *and* $\ell \nsubseteq \Delta$.

Imposing the τ-reality condition $\overline{g(\bar\lambda)} = g(\lambda)$ on both types, we obtain that one pole, say α, has to be real and so is the residue A there. It is convenient in this case to let ℓ denote the real line $\mathbb{R}\cdot(a, b, 1)$ in $\mathbb{R}^3$ and let Δ_0 denote $\Delta \cap \mathbb{R}^3$. We then have:

Corollary 4.4. *The simplest rational element in* $\Lambda_-^{\tau,\sigma}\,\mathrm{GL}(3,\mathbb{C})$ *is either* $g_{\alpha,\ell}$ *of rank* 1 *type or* $h_{\alpha,\ell}$ *of rank* 2 *type, where* $\alpha \in \mathbb{R}_*$ *and the real line* $\ell \nsubseteq \Delta_0$.

We are ready to compute the dressing action of these simple elements.

Lemma 4.5. *Let* $g_{\alpha,\ell} \in \Lambda_-^{\tau,\sigma}\,\mathrm{GL}(3,\mathbb{C})$ *as in* (4.7)*, and* $F \in \Lambda_+^{\tau,\sigma}\,\mathrm{SL}(3,\mathbb{C})$*. If* $\tilde{\ell} := \ell F(\alpha) \notin \Delta_0$*, then* $g_{\alpha,\ell} \cdot F$ *can be factored uniquely as*

$$g_{\alpha,\ell} \cdot F = \tilde{F} \cdot g_{\alpha,\tilde{\ell}} \ \in\ \Lambda_+^{\tau,\sigma}\,\mathrm{SL}(3,\mathbb{C}) \times \Lambda_-^{\tau,\sigma}\,\mathrm{GL}(3,\mathbb{C}).$$

Proof. It suffices to prove that $\tilde{F} := g_{\alpha,\ell} \cdot F \cdot g_{\alpha,\tilde{\ell}}^{-1}$ lies in $\Lambda_+^{\tau,\sigma}\,\mathrm{SL}(3,\mathbb{C})$. Since $\tilde{F}$ satisfies the reality conditions (4.2) and is holomorphic in $\mathbb{C}_*$ except for possible simple poles coming from the poles of $g_{\alpha,\ell}$ and $g_{\alpha,\tilde{\ell}}^{-1}$, we only need to prove that the residues of $\tilde{F}$ are zero at both α and $-\alpha$. But

$$\mu(\ g_{\alpha,\ell}(\lambda)\ ) = g_{\alpha,\ell}(-\lambda) \iff P = g_{\alpha,\ell}(\lambda)\ P\ g_{\alpha,\ell}(-\lambda)^t\ ,$$

whose residue is zero at α implies that $\ell P g_{\alpha,\ell}(-\alpha)^t = 0$, or equivalently $g_{\alpha,\ell}(-\alpha) P \ell^t = 0$. These two equations are also true for $\tilde{\ell}$. Therefore $(a,b,1)F(\alpha) \in \tilde{\ell}$ and the special form of A in (4.6) imply that

$$\mathrm{Res}_\alpha\, \tilde{F} = 2\alpha\ A\ F(\alpha)\ P\ g_{\alpha,\tilde{\ell}}(-\alpha)^t\ P = 0,$$

and $F(-\alpha)P\,(\tilde{a},\tilde{b},1)^t \in [F(-\alpha)PF(\alpha)^t]\,\ell^t = P\ell^t$ implies that

$$\mathrm{Res}_{-\alpha}\, \tilde{F} = -2\alpha\ g_{\alpha,\ell}(-\alpha)\ F(-\alpha)\ P\ \tilde{A}^t\ P = 0.$$

The proof is completed once we notice that $\det \tilde{F} = 1$ by Remark 4.2. ☐

Theorem 4.6. *The dressing action of rank* 1 *type* $g_{\alpha,\ell}$ *on the affine frames* $F(x,y,\lambda)$ *of proper indefinite affine spheres gives the classical Tzitzéica transformation, provided an open condition that* $\ell F(x,y,\alpha) \notin \Delta_0$*. The dressing action of rank* 2 *type* $h_{\alpha,\ell}$ *gives the dual transformation.*

Proof. By Lemma 4.5, for fixed (x,y), we have the factorization

$$g_{\alpha,\ell}(\lambda) \cdot F(x,y,\lambda) = \tilde{F}(x,y,\lambda) \cdot g_{\alpha,\tilde{\ell}}(\lambda)$$

with $\tilde{\ell} = \ell F(x,y,\alpha)$. From $F(x,y,\alpha) = ((X_\alpha)_x/\alpha,\ \alpha(X_\alpha)_y/h,\ X_\alpha)$, we get

$$(a,b,1)\ F(x,y,\alpha) = (\ \phi_x/\alpha,\ \alpha\phi_y/h,\ \phi\ ) \in \tilde{\ell},$$

where $\phi := (a,b,1)X_\alpha$ is a scalar solution of (2.9) with parameter α. Note that a constant scaling of ϕ does not change Tzitzéica transformation (2.10), and the solution space of the linear system (2.9) has dimension 3. Therefore, by varying ℓ, ϕ can be generic scalar solution up to a constant multiple.

By the discussion at the beginning of this section, the third column of $\tilde{F}$ produces new affine sphere, so does the affine transformation of it by $g_{\alpha,\ell}^{-1}$:

$$\begin{aligned}
\hat{X} &:= g_{\alpha,\ell}^{-1}(\tilde{F})_3 \\
&= \big(F(\lambda)\ g_{\alpha,\tilde{\ell}}(\lambda)^{-1}\big)_3 \\
&= \big(F(\lambda)\ P\ g_{\alpha,\tilde{\ell}}(-\lambda)^t\ P\big)_3 \\
&\overset{(4.7)}{=} (\ X_x/\lambda,\ \lambda X_y/h,\ X)\ \cdot\ \frac{1}{\lambda^3+\alpha^3}\ \cdot\ \begin{pmatrix} 2\alpha^3\lambda\phi_y/(h\phi) \\ -2\lambda^2\phi_x/\phi \\ \lambda^3-\alpha^3 \end{pmatrix} \\
&= \frac{(\lambda^3-\alpha^3)hX - 2\lambda^3(\ln\phi)_x X_y + 2\alpha^3(\ln\phi)_y X_x}{(\lambda^3+\alpha^3)h}\ .
\end{aligned}$$

The corresponding solution to Tzitzéica equation is given by

$$\hat{h} = \hat{X}_{xy}/\hat{X} = h - 2(\ln\phi)_{xy}.$$

This is exactly the classical Tzitzéica transformation (2.10) with

$$\gamma = \lambda^3, \qquad \gamma_1 = \alpha^3, \qquad \phi_1 = \phi.$$

In rank 2 type case, there is a similar factorization $h_{\alpha,\ell}\cdot F = \tilde{F}\cdot h_{\alpha,\tilde{\ell}}$ when $\tilde{\ell} := \mathrm{Kernel}^t(AF(\alpha)P) \nsubseteq \Delta_0$. We omit the details and present the corresponding transformation on affine spheres:

$$\tilde{X} = \frac{(\lambda^3+\alpha^3)hX - 2\lambda^3(\ln\phi)_x X_y - 2\alpha^3(\ln\phi)_y X_x}{(\lambda^3-\alpha^3)h},$$

where ϕ is the same scalar solution as the rank 1 type case. We see that

$$\hat{X}_\lambda = -\tilde{X}_{-\lambda}\ ,$$

i.e., $-\tilde{X}$ gives the dual of $\hat{X}$. This completes the proof. □

5. Permutability theorem and complex Tzitzéica transformations

Let us briefly review the classical description of the permutability theorem. In Theorem 2.3, let ϕ_1, ϕ_2 be the scalar solution of (2.9) with parameter γ_1, γ_2 respectively. Then using ϕ_1 to apply Tzitzéica transformation on h, we get a new solution to Tzitzéica equation:

$$h_1 := h - 2(\ln\phi_1)_{xy}\ .$$

Applying Tzitzéica transformation (2.10) to (ϕ_2, γ_2), we obtain

$$\phi_{12} := \frac{(\gamma_2 - \gamma_1)h\phi_2 - 2\gamma_2(\ln\phi_1)_x(\phi_2)_y + 2\gamma_1(\ln\phi_1)_y(\phi_2)_x}{(\gamma_2 + \gamma_1)h} \tag{5.1}$$

as a scalar solution to (2.9) with new h_1 and parameter γ_2. Therefore we can use ϕ_{12} to apply Tzitzéica transformation again on the new h_1, i.e.,

$$h_{12} = h_1 - 2(\ln\phi_{12})_{xy}$$

will give another solution to Tzitzéica equation. In this two step iteration, we may interchange the roles of ϕ_1 and ϕ_2 to obtain h_{21} as another new solution. The permutability theorem claims $h_{12} = h_{21}$. Similarly we can apply this two step iteration to the affine sphere X to obtain X_{12} and X_{21} respectively, and the equality $X_{12} = X_{21}$ still holds.

We will give a group point of view to this permutability theorem.

Lemma 5.1. *Let $g_{\alpha_i,\ell_i}(\lambda)$ $(i = 1, 2)$ be of rank 1 type with $\alpha_1^3 \neq \pm\alpha_2^3$. If both $\tilde{\ell}_1 := \ell_1 g_{\alpha_2,\ell_2}(\alpha_1)^{-1}$ and $\tilde{\ell}_2 := \ell_2 g_{\alpha_1,\ell_1}(\alpha_2)^{-1}$ are not in Δ, then*

$$g_{\alpha_2,\tilde{\ell}_2}\ g_{\alpha_1,\ell_1} = g_{\alpha_1,\tilde{\ell}_1}\ g_{\alpha_2,\ell_2}\ . \tag{5.2}$$

Proof. It is equivalent to prove that $f := g_{\alpha_1,\tilde{\ell}_1} g_{\alpha_2,\ell_2} g_{\alpha_1,\ell_1}^{-1}$ equals $g_{\alpha_2,\tilde{\ell}_2}$. First of all, they both are rational elements in the group $\Lambda_-^\sigma\,\mathrm{GL}(3,\mathbb{C})$. It suffices to prove that their poles and residues are the same.

Let $l_i = (a_i, b_i, 1)$ and $\tilde{l}_i = (\tilde{a}_i, \tilde{b}_i, 1)$ span ℓ_i and $\tilde{\ell}_i$ respectively. Similar to the proof of Lemma 4.5, we compute that

$$\mathrm{Res}_{\alpha_1} f = 2\alpha_1\ \widetilde{A}_1\ g_{\alpha_2,\ell_2}(\alpha_1)\ P\ g_{\alpha_1,\ell_1}(-\alpha_1)^t\ P = 0$$

since $\tilde{l}_1 g_{\alpha_2,\ell_2}(\alpha_1) \in \ell_1$, and

$$\mathrm{Res}_{-\alpha_1} f = -2\alpha_1\ g_{\alpha_1,\tilde{\ell}_1}(-\alpha_1)\ g_{\alpha_2,\ell_2}(-\alpha_1)\ P\ A_1^t\ P = 0$$

since $g_{\alpha_2,\ell_2}(-\alpha_1) P\, l_1^t \in [g_{\alpha_2,\ell_2}(-\alpha_1)\, P\, g_{\alpha_2,\ell_2}(\alpha_1)^t]\, \tilde{\ell}_1^t = P\, \tilde{\ell}_1^t$. Thus f has only three simple poles $\{\alpha_2, \epsilon^2\alpha_2, \epsilon^4\alpha_2\}$, same as $g_{\alpha_2,\tilde{\ell}_2}$.

Now due to the ν reality condition in (4.2), we only need to prove that their residues at α_2 are the same. But

$$\begin{aligned}
\mathrm{Image}\left(\mathrm{Res}_{\alpha_2} f^t\right) &= \mathrm{Image}\left(\left(g_{\alpha_1,\ell_1}(\alpha_2)^{-1}\right)^t A_2^t\ g_{\alpha_1,\tilde{\ell}_1}(\alpha_2)^t\right) \\
&= \left(g_{\alpha_1,\ell_1}(\alpha_2)^{-1}\right)^t \ell_2^t \\
&= \tilde{\ell}_2^t \quad = \quad \mathrm{Image}\left(\mathrm{Res}_{\alpha_2} g^t_{\alpha_2,\tilde{\ell}_2}\right).
\end{aligned}$$

Then $\mathrm{Res}_{\alpha_2} f$ must be the same as $\mathrm{Res}_{\alpha_2}(g_{\alpha_2,\tilde{\ell}_2})$ since rank 1 type residue is uniquely determined by the above image. This completes the proof. ∎

Example 5.2. Choose two nonzero poles α_1, α_2 such that $\alpha_1^3 \neq \pm\alpha_2^3$, and let $\ell_i = \mathbb{C}\cdot(0, b_i, 1)$ for $i = 1, 2$. Then $g_{\alpha_2,\tilde{\ell}_2}\ g_{\alpha_1,\ell_1} = g_{\alpha_1,\tilde{\ell}_1}\ g_{\alpha_2,\ell_2}$ holds for $\tilde{\ell}_i = \mathbb{C}\cdot(0, \tilde{b}_i, 1)$ with

$$\tilde{b}_1 = \frac{(\alpha_1^3+\alpha_2^3)b_1 - 2\alpha_1\alpha_2^2 b_2}{(\alpha_1^3-\alpha_2^3)}, \qquad \tilde{b}_2 = \frac{2\alpha_1^2\alpha_2 b_1 - (\alpha_1^3+\alpha_2^3)b_2}{(\alpha_1^3-\alpha_2^3)}.$$

Theorem 5.3. *Use the same notation and the factorization formula (5.2) in Lemma 5.1. Let all α_i, ℓ_i be real. Let $F_1 := g_{\alpha_1,\ell_1} * F$ and $F_2 := g_{\alpha_2,\ell_2} * F$, where $*$ is the dressing action on the frames $F(x, y, \lambda)$ of affine spheres. Then the following holds and implies the classical permutability theorem:*

$$\begin{aligned} &(g_{\alpha_2,\tilde{\ell}_2} g_{\alpha_1,\ell_1}) * F = g_{\alpha_2,\tilde{\ell}_2} * F_1 \quad (\ =: F_{12}\ ) \\ =\ &(g_{\alpha_1,\tilde{\ell}_1} g_{\alpha_2,\ell_2}) * F = g_{\alpha_1,\tilde{\ell}_1} * F_2 \quad (\ =: F_{21}\ ). \end{aligned} \tag{5.3}$$

Proof. Because the dressing action is a group action, (5.3) certainly holds by (5.2). Let $l_i = (a_i, b_i, 1)$ and $\tilde{l}_i = (\tilde{a}_i, \tilde{b}_i, 1)$ span ℓ_i and $\tilde{\ell}_i$ respectively. From the proof of Theorem 4.6, $F_1 := g_{\alpha_1,\ell_1} * F$ means in classical terms the following: Tzitzéica transformation via $\phi_1 := l_1(F(\alpha_1))_3$ on $X = (F)_3$ gives a new affine sphere $X_1 := g^{-1}_{\alpha_1,\ell_1}(F_1)_3$. Therefore $F_{12} := g_{\alpha_2,\tilde{\ell}_2} * F_1$ implies that Tzitzéica transformation via $\phi_{12} := \tilde{l}_2(F_1(\alpha_2))_3$ on $(F_1)_3 = g_{\alpha_1,\ell_1} X_1$ gives a new affine sphere $g^{-1}_{\alpha_2,\tilde{\ell}_2}(F_{12})_3$. We observe that

$$\phi_{12} := \tilde{l}_2(F_1(\alpha_2))_3 = c_0 l_2\, g_{\alpha_1,\ell_1}(\alpha_2)^{-1}(F_1(\alpha_2))_3 = c_0 l_2\, X_1(\alpha_2),$$

which coincides with the classical formula (5.1) except for a negligible constant c_0 when we plug in $\phi_2 := l_2\, X(\alpha_2)$.

So Tzitzéica transformation via ϕ_{12} on $X_1 = g^{-1}_{\alpha_1,\ell_1}(F_1)_3$ produces

$$X_{12} = g^{-1}_{\alpha_1,\ell_1}\left(g^{-1}_{\alpha_2,\tilde{\ell}_2} F_{12}\right)_3 = \left(g^{-1}_{\alpha_1,\ell_1} g^{-1}_{\alpha_2,\tilde{\ell}_2} F_{12}\right)_3 .$$

Similarly $X_{21} = \left(g^{-1}_{\alpha_2,\ell_2} g^{-1}_{\alpha_1,\tilde{\ell}_1} F_{21}\right)_3$. Therefore, by (5.2) and (5.3), we obtain the classical permutability theorem: $X_{12} = X_{21}$, which automatically implies $h_{12} = h_{21}$ for the corresponding affine metrics. $\square$

There is some rational element in $\Lambda_-^{\tau,\sigma}\,\mathrm{GL}(3,\mathbb{C})$ which has 6 simple poles but none of them are real. The poles must form two conjugate triples as $\{\alpha, \epsilon^2\alpha, \epsilon^4\alpha\}$ and $\{\bar{\alpha}, \epsilon^2\bar{\alpha}, \epsilon^4\bar{\alpha}\}$, where we may assume $0 < \arg(\alpha) < \pi/3$ without loss of generality. So such element is not product of *real* rank 1 or 2 type elements. In fact, we can use Lemma 5.1 to construct them:

Proposition 5.4. *Let* $\alpha \in \mathbb{C}_*$ *with* $\arg(\alpha) \in (0, \pi/6) \cup (\pi/6, \pi/3)$*. Let* $\ell \not\subseteq \Delta$*. If* $\ell^* := \bar{\ell} \cdot g_{\alpha,\ell}(\bar{\alpha})^{-1} \not\subseteq \Delta$*, then* $f_{\alpha,\ell} := g_{\bar{\alpha},\ell^*}\, g_{\alpha,\ell} \in \Lambda_-^{\tau,\sigma}\,\mathrm{GL}(3,\mathbb{C})$*.*

Proof. We first observe that $f_{\alpha,\ell}$ lies in $\Lambda_-^{\sigma}\,\mathrm{GL}(3,\mathbb{C})$. It suffices to verify $\overline{f(\bar{\lambda})} = f(\lambda)$, which is

$$g_{\alpha,\overline{\ell^*}}\; g_{\bar{\alpha},\bar{\ell}} \;=\; g_{\bar{\alpha},\ell^*}\; g_{\alpha,\ell}.$$

Since $\ell^* = \bar{\ell} \cdot g_{\alpha,\ell}(\bar{\alpha})^{-1} \not\subseteq \Delta$ implies $\overline{\ell^*} = \ell \cdot g_{\bar{\alpha},\bar{\ell}}(\alpha)^{-1} \not\subseteq \Delta$ and $\arg(\alpha) \neq \pi/6$ implies $\alpha^3 \neq -\bar{\alpha}^3$, the above factorization holds by Lemma 5.1. □

The dressing action of $f_{\alpha,\ell}$ on affine spheres can be viewed as the composition of two 'conjugate' complex Tzitzéica transformations, which produces a real solution in the end. The Permutability Theorem 5.3 can be applied to compute this action. Solutions from this construction are often called breather type solutions.

It is not hard to show, by a similar residue calculus as before, that any rational element with 6 simple poles as above can be constructed from Proposition 5.4. What would be much messier, if not harder, to prove is that the subgroup of all rational elements in $\Lambda_-^{\tau,\sigma}\,\mathrm{GL}(3,\mathbb{C})$ is generated by $g_{\alpha,\ell}$'s, $f_{\alpha,\ell}$'s, and their rank 2 type brothers. This subgroup can then be regarded as the group of Tzitzéica transformations on affine spheres. We will leave this interesting problem for future study.

6. Basic examples

In this section, we use x, y, z as the standard $\mathbb{R}^3$ coordinates to represent the immersion X, and use u, v to denote the asymptotic coordinates of the affine spheres.

Example 6.1 (The **vacuum** solution)**.** The vacuum solution to Tzitzéica equation (see also [6], [15]) is $\omega_0 \equiv 0$ (or $h_0 \equiv 1$). One can integrate (3.1) to obtain the whole family of frames. The Cartesian equation of the surface is then obtained by the determinant:

$$x^3 + y^3 + z^3 - 3xyz = 1.$$

Note that it is independent of the parameter λ. So this family is really a family of parametrizations of the same affine sphere.

A general scalar solution of system (2.9) with parameter $\gamma = \lambda^3$ is:

$$\phi(\lambda) = c_0 R(\lambda) + c_1 R(\epsilon^2\lambda) + c_2 R(\epsilon^4\lambda), \tag{6.1}$$

where $R(\lambda) := \exp(\lambda u + \lambda^{-1} v)$.

Therefore we may choose the following asymptotic parametrizations of the vacuum affine sphere after certain affine transformation:

$$X_0(u,v,\lambda) = \begin{pmatrix} \exp[-(\lambda u + \lambda^{-1}v)/2]\ \cos[\sqrt{3}(\lambda u - \lambda^{-1}v)/2] \\ \exp[-(\lambda u + \lambda^{-1}v)/2]\ \sin[\sqrt{3}(\lambda u - \lambda^{-1}v)/2] \\ \frac{2}{3\sqrt{3}}\ \exp(\lambda u + \lambda^{-1}v) \end{pmatrix},$$

which is a surface of revolution (Note that Jonas has classified all affine spheres of revolution in [10] using elliptic functions).

Example 6.2 (The **one-soliton** solution)**.** Apply Tzitzéica transformation to the vacuum solution we obtain the one-soliton solution h_1. By (6.1), $\phi_1 = \phi(\lambda_1)$ is a scalar solution of system (2.9) with parameter $\gamma_1 = \lambda_1^3$. It is real when $c_0 \in \mathbb{R}$, $c_1 = \overline{c_2}$, and $\lambda_1 \in \mathbb{R}_*$. Compute $h_1 = 1 - 2(\ln\phi_1)_{uv}$:

$$h_1 = 1 - \frac{6\beta_0 \exp(3s_1/2)\cos\left(\sqrt{3}\ t_1/2 + \theta_0\right) + 1.5}{\left[\beta_0\exp(3s_1/2) + \cos\left(\sqrt{3}\ t_1/2+\theta_0\right)\right]^2},$$

where $c_1 = \rho_0 e^{\mathrm{i}\theta_0}$, $\beta_0 = c_0/(2\rho_0)$, $s_1 = \lambda_1 u + \lambda_1^{-1}v$, and $t_1 = \lambda_1 u - \lambda_1^{-1}v$. The family of affine apheres $X_1(u,v,\lambda)$ has a long expression given by (2.10).

When $\beta_0 = 0$ (i.e. $c_0 = 0$), we have the special solution:

$$h_1 = 1 - 1.5\sec^2\left[\sqrt{3}\ (\lambda_1 u - \lambda_1^{-1}v)/2 + \theta_0\right]. \tag{6.2}$$

We give explicit formula for this family of affine spheres:

$$X_1(u,v,\lambda) = \frac{(\lambda^3-\lambda_1^3)}{(\lambda^3+\lambda_1^3)}X_0(u,v,\lambda) + \frac{\sqrt{3}\lambda\lambda_1\tan\left(\sqrt{3}t_1/2+\theta_0\right)}{(\lambda^3+\lambda_1^3)}\cdot$$
$$\begin{pmatrix} e^{-\frac{s}{2}}\left[\lambda\cos\left(\sqrt{3}t/2+4\pi/3\right) + \lambda_1\cos\left(\sqrt{3}t/2+2\pi/3\right)\right] \\ e^{-\frac{s}{2}}\left[\lambda\sin\left(\sqrt{3}t/2+4\pi/3\right) + \lambda_1\ \sin\left(\sqrt{3}t/2+2\pi/3\right)\right] \\ 2e^{s}(\lambda+\lambda_1)/(3\sqrt{3}) \end{pmatrix}. \tag{6.3}$$

Note that ϕ_1 need not be real to produce real h_1. For example, (6.2) will be a real hyperbolic function solution when λ_1 and θ_0 are pure imaginary. In this case the real (or imaginary) part of (6.3) still produces affine spheres, among which are the stationary and traveling one-soliton affine sphere shown in [16]. Some pictures have already been shown in [15] and [16]. In terms of dressing action, λ_1 is the pole of some rank 1 type simple element $g_{\lambda_1,\ell}$. So dressing actions of $g_{\lambda_1,\ell}$ with a pure imaginary pole (or a pole whose argument is $\pm\pi/6$) may also produce new real affine spheres sometime.

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